WHEN IS A GROUNDING 'GOOD', FOR WHOM, AND HOW CAN WE BUILD THEM?

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What is it that makes a mathematical form a good grounding for a mathematical discipline? This is a question that has been extensively discussed in cognitive science over the last decade (Barsalou, 1999, 2008; Lakoff & Nuñez, 2000). Intuitively, some mathematical forms seem to serve as very good 'groundings' of a mathematical domain. For instance, the number line has been shown to be useful for students attempting to understand numerical magnitude (Siegler, Thompson & Schneider, 2011; Thompson & Schneider, 2010); negatives as reflections (Tsang et al., in press), basic arithmetic operations (e.g., addition and subtraction as rightward and leftward motion; multiplication and division as scaling). The unit circle has been seen to be useful for grounding trigonometric identities (Mickey & McClelland, 2013). Multiplication has many intuitively 'good' grounds, including repeated addition, area, and scaling.

It has been argued that good groundings occur when the abstractions of mathematics are made meaningful, and that this happens when abstractions are situated in everyday physical activities (Lakoff & Nuñez, 2000; Braithwaite & Goldstone, 2013), and in visuo-spatial—as opposed to formal, systems (e.g., Mickey & McClelland, 2013). For instance, the number line can be further grounded as beads on a string. Linear functions can be grounded in terms of constant motions (Nathan, Kintsch & Young, 1992). Although the trigonometric functions are uniquely specified by the definitions

$C(x-y)=C(x)C(y)+S(x)S(y)$ and

$S(x-y)=S(x)C(y)-C(x)S(y)$

With x ,y in R and C, S not identically zero (Robison, 1968; this characterization due to Steven Taschuk), it seems that the sine and cosine functions become meaningful when interpreted as the coordinates of points on the unit circle (or even more the signed height and width of inscribed triangles).

Symbol systems, on this account are not good groundings because they are arbitrary (rather than meaningful), non-spatial, and unrelated to everyday physical experience. On this hypothesis, good groundings are 'embodied' or 'situated' groundings.

In my talk, I'll suggest instead that symbol structures and classical diagrams are the same sort of creature, and that 'embodiment' is at best correlated with the features that make for good groundings. In particular, I'll argue that most or all mathematical models (like most or all mental models) comprise a depicted spatial structure and a deontic structure—a set of appropriate and inappropriate behaviors (Wittgenstein, 1978). This level of description unifies spatial symbol systems such as the algebra of arithmetic equations and other diagrams such as function graphs, networks, or the unit circle. It is worth emphasizing that on this account of grounding, the degree of grounding is a function of the state of the reasoner. Symbol systems can be good groundings for those steeped in them; embodiment provides routinely good groundings only because physical experiences tend to be widely shared.

A common proposal for differentiating diagrams from symbol systems is that diagrams may be diagrammatically—that they may carry meaning 'directly', while symbol systems carry meaning only indirectly or through interpretation (Stenning, 2000). This distinction is real, but it is continuous, not categorical, across systems of representations. There are, in all mathematical models, two sorts of meaning: one sort inheres in the system; this is the sense by which symbol systems are deeply meaningful, in that they written inscriptions depict the spatial structure of some (imagined) physical situation—a situation made of symbols. This is also the sense at work in Cantor's diagonalization proof, in which numerals are taken themselves as objects. Another sense of meaning is referential, in which a mathematical model is taken to stand in for another situation, and is aligned with that situation. Both traditional diagrams and symbols systems may be used either diagrammatically or referentially, and meaning is built out of both through combinations of reasoning about surface form and referential meaning (Landy, Allen & Zednik, 2014).

So then what does make some models so amazingly good at grounding understandings of broad domains? I'll suggest that this comes down not to some categorical or qualitative distinction to be drawn on the basis of the semiotic properties of the sign systems, but to good old-fashioned cognitive constraints and limitations. A mental model is good when the depicted space (mediated by an inscriptive practice) is easy to remember and reconstruct, the deontology is simple to remember and easy to physically instantiate, and the collection of important inferences are easy to draw within capacity limitations and using available or easily adapted perceptual-motor routines. A situation or mathematical model serves as a good grounding for another domain when the other domain does not have these features, the grounding does, and the mapping between the two is easy to traverse. Symbolic systems are poor groundings because of the ways they mismatch the cognitive state of the learner, and therefore carry high memory burdens, many opportunities for error, and few opportunities for inference.

That everyday physical experiences are often good groundings can now be seen to be a contingent, not a constitutive, characteristic. It is often the case that we have already adapted our perceptual-motor processing to the deontic requirements of everyday situations, that we are familiar with many inferences about them that can be readily exported, and that we are familiar enough to be able to reconstruct them with high regularity. This, and not any deeper notion of 'embodiment', is what makes everyday experience so often a good conceptual grounding for higher mathematics. This perspective unifies the kind of grounding a novice gets from understanding 'more' as 'rightward on the line', or 'rightward on this string of beads', and the kind of grounding an algebraic topologist gets from identifying deformations of topological spaces in high dimensions with abstract algebraic groups.

Once we have seen good grounding for what it is, a function of cognitive economy and useful prior experience, we can then design symbolic systems that are more effective—that have lower load, that yield better generative and predictive models for making inferences, that are easily exported, and that better align with pre-existing implementation systems in human reasoners. I'll present one such attempt, the 'Graspable Mathematics' system, and show how this dynamical algebra potentially lowers the burden on the algebraic reasoner, improving the intrinsic meaningfulness of the mathematical model and better grounding other mathematical models.

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